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The Equation $(g(x))_x Ax - Bg(x) = h(x)$

H. K. WIMMER*

Mathematisches Institut, Universität Würzburg, D 8700 Würzburg, West Germany

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1. A PROBLEM ON INTEGRAL MANIFOLDS

We consider a system of the form

$$\begin{aligned} \dot{x} &= Ax + p_1(x, y) \\ \dot{y} &= By + p_2(x, y) \end{aligned} \tag{1}$$

with $x \in \mathbb{R}^k$, $y \in \mathbb{R}^m$ and A and B real constant matrices of appropriate size. It is assumed that in a neighbourhood of $(0, 0)$ p_1 and p_2 have partial derivatives of arbitrarily high order ($p_i \in C^\infty$) and

$$p_i = 0, \quad (p_i)_x = 0, \quad (p_i)_y = 0 \quad \text{for} \quad (x, y) = (0, 0) \quad \text{and} \quad i = 1, 2.$$

If there is a function $s = s(x) = (s_1(x),\ldots, s_m(x))^T$, $s \in C^\infty$, such that $M = \{(t, x, y) \mid y = s(x);\ t, x \text{ arbitrary}\}$ is an integral manifold of (1), then s satisfies the partial differential equation

$$(s(x))_x[Ax + p_1(x, s(x))] = Bs(x) + p_2(x, s(x)) \tag{2}$$

where s_x is the Jacobian of s. Let s have the Taylor expansion $s(x) = \sum_{j=1}^{\infty} s^j(x)$ at $x = 0$. The components $s_i^j(x)$ of the vector $s^j(x) = (s_1^j(x),\ldots, s_m^j(x))^T$ are polynomials in $x_1,\ldots, x_k$ and homogeneous of degree j. Because of (2) the functions $s^j(x)$ are related recursively by

$$(s^j(x))_x\, Ax = Bs^j(x) + q^j(x), \tag{3}$$

where q^j is homogeneous of degree j and depends on $s^1,\ldots, s^{j-1}$ (see [2, Chapter V.8]).

Using stability arguments it is shown in [2] that (3) has a unique solution $s^j(x)$ for any $q^j(x)$, if A is stable and all eigenvalues of B have positive real parts.—In the first part of this note we study

$$(g(x))_x\, Ax - Bg(x) = h(x) \tag{4}$$

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as a linear—algebraic problem and prove a necessary and sufficient condition for the solvability of (4). We then discuss an equation which is related to integral manifolds of periodic equations of type (1).

2. The Polynomial Equation

Let $\mathbb{C}_r[x]$ be the linear space of all complex polynomials in $x_1, \dots, x_k$ which are homogeneous of degree r. In this section we focus on a linear operator $A^{\{r\}}\colon \mathbb{C}_r[x] \to \mathbb{C}_r[x]$, which is defined for each $A \in \mathbb{C}^{k\times k}$ by

$$A^{\{r\}}\, q(x) = [\operatorname{grad} q(x)]^T Ax.$$

The fact that $\mathbb{C}_r[x]$ is isomorphic to a space V of symmetric tensors of order r will clarify the structure of $A^{\langle r\rangle}$. Instead of (4) we shall deal with an equation in V.

We need the following definitions and remarks for which we refer to [1] and [5]. The Kronecker product of $D = (d_{\mu\nu}) \in R^{m\times n}$ and $F \in R^{s\times t}$, where R is a ring, is defined to be the $ms \times nt$ block matrix

$$D \otimes F = (d_{\mu\nu} F); \qquad \mu = 1,\dots, m, \quad \nu = 1,\dots, n. \tag{5}$$

If $x = (x_1, \dots, x_k)^T$ is a vector with indeterminates over $\mathbb{C}$ we define

$$\bigotimes^r x = \underbrace{x \otimes \cdots \otimes x}_{r \text{ times}}.$$

For a matrix $\Phi = Pe^R$ the r-th Kronecker power $\bigotimes^r \Phi$ is

$$\begin{aligned}\bigotimes^r \Phi &= \left(\bigotimes^r P\right)\left(\bigotimes^r e^R\right)\\ &= \left(\bigotimes^r P\right) \exp\left(R \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes R\right).\end{aligned}$$

In the set of r-tuples

$$\Gamma := \{(\gamma_1, \dots, \gamma_r) \mid \gamma_i \in \mathbb{N},\ 1 \leqslant \gamma_i \leqslant k,\ i = 1,\dots, r\}$$

we call $\gamma = (\gamma_1, \dots, \gamma_r)$ and $\omega = (\omega_1, \dots, \omega_r)$ equivalent, $\gamma \sim \omega$, if there is a permutation $\pi \in S_r$ such that $(\omega_1, \dots, \omega_r) = (\gamma_{\pi 1}, \dots, \gamma_{\pi r})$.—Let $\bigotimes^r \mathbb{C}^k$ be the r-th tensorial power of $\mathbb{C}^k$. If e_i is the i-th unit vector in $\mathbb{C}^k$, $e_i = (0,\dots, 0, 1, 0,\dots, 0)^T \in \mathbb{C}^k$, $i = 1,\dots, k$, then an arbitrary tensor $y \in \bigotimes^r \mathbb{C}^k$ has the form

$$y = \sum_{(\omega_1,\dots,\omega_r)\in\Gamma} c_{(\omega_1,\dots,\omega_r)} e_{\omega_1} \otimes \cdots \otimes e_{\omega_r}. \tag{6}$$

The product $\otimes$ in (6) is given by (5) and y is an rk-vector. y is a symmetric tensor, if $\omega \sim y$ implies $c_\omega = c_y$. The subspace of symmetric tensors of $\otimes^r \mathbb{C}^k$ is denoted by $V^r\mathbb{C}^k$ and $d := \dim V^r\mathbb{C}^k = \binom{k+r-1}{r}$. We sometimes set $V := V^r\mathbb{C}^k$.

Each polynomial $q(x) \in \mathbb{C}_r[x]$ can be written in a unique way as

$$q(x) = \sum_{\omega=(\omega_1,\ldots,\omega_r)\in\Gamma} c_\omega x_{\omega_1} \cdots x_{\omega_r}$$

such that $c_\omega = c_y$, if $\omega \sim \gamma$. Let $q = (\cdots, c_\omega, \cdots)^T \in \otimes^r \mathbb{C}^k$ contain the coefficients of $q(x)$. If the elements of q are ordered corresponding to the lexicographical ordering of Γ, then q is in $V^r\mathbb{C}^k$ and

$$q(x) = q^T \left(\bigotimes^r x\right). \tag{7}$$

The mapping $q(x) \mapsto q$ is an isomorphism of $\mathbb{C}_r[x]$ onto $V^r\mathbb{C}^k$ [1, p. 203]. c_ω will be called the ω-entry of q.

The following theorem is due to Ljapunov. We give a new proof which shows that the operator to $A^{\{r\}}$ is a partial derivation.

THEOREM 1[4]. *Let $A \in \mathbb{C}^{k\times k}$ have the eigenvalues $\lambda_1, \ldots, \lambda_k$. Then the linear operator $A^{\{r\}}: \mathbb{C}_r[x] \to \mathbb{C}_r[x]$ which is defined by*

$$A^{\{r\}} q(x) = [\operatorname{grad} q(x)]^T Ax$$

has the eigenvalues $r_1\lambda_1 + r_2\lambda_2 + \cdots + r_k\lambda_k$ with integers $r_1 \geqslant 0, \ldots, r_k \geqslant 0$ and $r_1 + r_2 + \cdots + r_k = r$.

Proof. From $q(x) = q^T(\otimes^r x)$ we get $[\operatorname{grad} q(x)]^T = q^T(I \otimes x \otimes \cdots \otimes x + \cdots + x \otimes \cdots \otimes x \otimes I)$. Therefore

$$\begin{aligned}[\operatorname{grad} q(x)]^T Ax &= q^T \left(A \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes A\right)\left(\bigotimes^r x\right) \\ &= q^T A^{\langle r\rangle} \left(\bigotimes^r x\right)\end{aligned}$$

where $A^{\langle r\rangle}: V^r\mathbb{C}^k \to V^r\mathbb{C}^k$ is defined by

$$A^{\langle r\rangle} := A \otimes I \otimes \cdots \otimes I + I \otimes A \otimes I \otimes \cdots \otimes I + \cdots I \otimes \cdots \otimes I \otimes A.$$

The operator $A^{\langle r\rangle}$ is a partial derivation on the space $V^r\mathbb{C}^k$ (see [5, p. 224]). Thus our assertion is contained in a general result on eigenvalues of partial derivations [5, p. 235–237]. In order to give a self contained proof we can argue directly as follows. For our purposes we can assume A to be upper triangular and the eigenvalue λ_i to be the (i, i)-entry of A. Then $A^{\langle r\rangle}$ is also upper triangular and for $\omega = (\omega_1, ..., \omega_r)$ the (ω, ω)-entry of $A^{\langle r\rangle}$ is

$$\lambda_{\omega_1} + \lambda_{\omega_2} + \cdots + \lambda_{\omega_r}. \tag{7}$$

$A^{\langle r\rangle}$ regarded as an operator on $\bigotimes^r \mathbb{C}^k$ has as its eigenvalues all the sums (7) with $\omega \in \Gamma$. If $A^{\langle r\rangle}$ is restricted to $V^r\mathbb{C}^k$, then its eigenvalues are again given by (7), but in this case ω varies over a complete set of representatives of equivalence classes of Γ.

Let $\mathbb{C}_r{}^m[x]$ be the linear space of all m-vectors $f(x) = (f_1(x), ..., f_m(x))^T$ with $f_i(x) \in \mathbb{C}_r[x]$, $i = 1, ..., m$. We write $(f(x))_x$ for the Jacobian of $f(x)$,

$$(f(x))_x = \left(\frac{\partial f_i}{\partial x_j}\right) = (\operatorname{grad} f_1(x), ..., \operatorname{grad} f_m(x))^T.$$

We define a linear space V^m of $m \times rk$ matrices by

$$V^m = \{W \mid W = (w_1, ..., w_m)^T, w_\mu \in V = V^r\mathbb{C}^k, \mu = 1, ..., m\}.$$

For each $f(x) \in C_r{}^m[x]$ there exists a unique matrix $F \in V^m$ such that $f(x) = F(\bigotimes^r x)$.

We now deal with the equation

$$(g(x))_x Ax - Bg(x) = h(x). \tag{4}$$

The assumptions on (4) are: $g(x)$, $h(x) \in \mathbb{C}_r{}^m[x]$, $A \in \mathbb{C}^{k\times k}$ and $B \in \mathbb{C}^{m\times m}$.—If G and $H \in V^m$ are such that $G(x) = G(\bigotimes^r x)$ and $h(x) = H(\bigotimes^r x)$, then (4) can be written as

$$GA^{\langle r\rangle}\left(\bigotimes^r x\right) - BG\left(\bigotimes^r x\right) = H\left(\bigotimes^r x\right)$$

which is equivalent to

$$GA^{\langle r\rangle} - BG = H. \tag{8}$$

It is known (see e.g. [3, p. 262]) that (8) has a solution G for any H, if and only if $\lambda^{\langle r\rangle} \neq \mu$ for all eigenvalues $\lambda^{\langle r\rangle}$ of $A^{\langle r\rangle}$ and all eigenvalues μ of B. Combining this observation with Theorem 1 we obtain the following result.

THEOREM 2. *The equation*

$$(g(x))_x Ax - Bg(x) = h(x)$$

has a unique solution $g(x) \in \mathbb{C}_r{}^m[x]$ *for any* $h(x) \in \mathbb{C}_r{}^m[x]$, *if and only if the eigenvalues* $\lambda_1, ..., \lambda_k$ *of* A *and the eigenvalues* $\mu_1, ..., \mu_m$ *of* B *satisfy the following conditions*:

$$r_1\lambda_1 + \cdots + r_k\lambda_k \neq \mu_j,$$
$$r_1 \geq 0, ..., r_k \geq 0, \qquad r_1 + \cdots + r_k = r; \qquad j = 1, ..., m.$$

3. A Periodic Equation

Let the system

$$\begin{aligned} \dot{x} &= A(t)\,x + p_1(t, x, y) \\ \dot{y} &= B(t)\,y + p_2(t, x, y) \end{aligned} \tag{9}$$

be given where the right hand sides are assumed to be periodic with period 1. In order to construct an integral manifold of (9) of the form $y = s(t, x)$ the differential equation

$$\frac{\partial}{\partial t} w(t, x) + \big(w(t, x)\big)_x A(t)\,x - B(t)\,w(t, x) = f(t, x) \tag{10}$$

has to be solved [2, p. 250].

Define $\mathbb{C}_r{}^m[x; t, 1]$ to be the space of all $f(t, x)$ such that $f(x, t) \in \mathbb{C}_r{}^m[x]$ for $t \in (-\infty, \infty)$, $f(t, x)$ is continous with respect to t and is 1-periodic, i.e. $f(t + 1, x) = f(t, x)$ for all t. Similarly set $\mathbb{C}^{p\times s}(t, 1)$ for all 1-periodic functions which map $\mathbb{R}$ continously into $\mathbb{C}^{p\times s}$. For $\mathbb{C}^{m\times 1}(t, 1)$ we write $\mathbb{C}^m(t, 1)$.

The assumptions on the equation (10) are: $A(t) \in \mathbb{C}^{k\times k}(t, 1)$, $B(t) \in \mathbb{C}^{m\times m}(t, 1)$, $f(t, x) \in \mathbb{C}_r{}^m[x; t, 1]$.—What are the conditions that (10) has a solution $w(t, x) \in C_r{}^m[x; t, 1]$ for any $f(t, x)$? Again based on stability arguments it is shown in [2] that a solution exists, if the characteristic multipliers of $\dot{x} = A(t)x$ and those of $\dot{y} = B(t)y$ are seperated by the unit circle. In this section we derive a necessary and sufficient solvability condions for (10).

Lemma 1 (see e.g. [2, p. 90]). *The equation* $\dot{x} = M(t)x + g(t)$, $M(t) \in \mathbb{C}^{m\times m}(t, 1)$, *has a solution* $x(t) \in \mathbb{C}^m(t; 1)$ *for any* $g(t) \in \mathbb{C}^m(t, 1)$, *if and only if the only solution* $x(t) \in \mathbb{C}^m(t, 1)$ *of*

$$\dot{x} = M(t)x \tag{11}$$

is $x(t) = 0$, *or equivalently if* $\mu = 1$ *is not a characteristic multiplier of* (11).

There is an obvious generalization of the preceding lemma to matrix differential equations.

Lemma 2. *Let the characteristic multipliers of* $\dot{v} = M(t)v$, $M(t) \in \mathbb{C}^{d\times d}(t, 1)$, *and those of* $\dot{y} = B(t)y$, $B(t) \in \mathbb{C}^{m\times m}(t, 1)$, *be* $\mu_1, \ldots, \mu_d$ *and* $\beta_1, \ldots, \beta_m$ *respectively. The equation*

$$\dot{W}(t) + W(t)\,M(t) - B(t)\,W(t) = F(t) \tag{12}$$

has a solution $W(t) \in \mathbb{C}^{m\times d}(t, 1)$ *for any* $F(t) \in \mathbb{C}^{m\times d}(t, 1)$, *if and only if*

$$\frac{\beta_j}{\mu_i} \neq 1, \qquad i = 1, \ldots, d, \quad j = 1, \ldots, m.$$

Proof. If the columns of the $m \times d$ matrix $W = (w_{ji})$ are stacked to an md-vector $\tilde{w} = (w_{11}, w_{21}, ..., w_{m1}, ..., w_{md})^T$, then the homogeneous equation $\dot{W} + WM - BW = 0$ can be transformed into a vector equation (see e.g. [3, p. 260])

$$\dot{\tilde{w}}(t) = (-M(t)^T \otimes I + I \otimes B(t))\, \tilde{w}(t). \tag{13}$$

Let $\Gamma(t) = K(t)\, e^{Lt}$ and $\Lambda(t) = Q(t)\, e^{St}$ be fundamental matrices of

$$\dot{v} = M(t)v \tag{14}$$

and

$$\dot{y} = B(t)y \tag{15}$$

respectively. Then

$$[\Gamma^{-1}(t)]^T \otimes \Lambda(t) = [(K^{-1}(t))^T \otimes Q(t)] \exp(-L^T \otimes I + I \otimes S)t$$

is a fundamental matrix of (13). The characteristic multipliers of (13) are the eigenvalues of $e^{-L^T} \otimes e^S$, that is the products of the eigenvalues of $(e^L)^{-1}$ and e^S which are given by $(\mu_i)^{-1} \beta_j$.

THEOREM 3. *Let the characteristic multipliers of $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ be $\alpha_1, ..., \alpha_k$ and $\beta_1, ..., \beta_m$ respectively. The equation* (10) *has a solution $w(t, x)$ in $\mathbb{C}_r^m[x; t, 1]$ for any $f(t, x) \in \mathbb{C}_r^m[x; t, 1]$, if and only if*

$$\alpha_1^{r_1} \alpha_2^{r_2} \cdots \alpha_k^{r_k} \neq \beta_j \tag{16}$$

where $r_1 \geqslant 0, ..., r_k \geqslant 0$, $r_1 + \cdots + r_k = r$, $j = 1, ..., m$.

Proof. We write $w(t, x)$ in (10) as $w(t, x) = W(t)(\otimes^r x)$ such that $W(t) \in V^m$ for all t. Similarly $f(t, x) = F(t)(\otimes^r x)$. Then (10) is equivalent to

$$\dot{W}(t) + W(t)\, M(t) - B(t)\, W(t) = F(t)$$

with $M(t) = A(t) \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes A(t)$. If $\Phi(t) = P(t) e^{Rt}$ is a fundamental matrix of $\dot{x} = A(t)x$, then

$$\otimes^r \Phi(t) = \left[\otimes^r P(t)\right] \left[\otimes^r e^{Rt}\right]$$

is a fundamental matrix of

$$\dot{v} = M(t)v, \qquad v(t) \in V^r\mathbb{C}^k. \tag{17}$$

The characteristic multipliers of (17) are

$$\alpha_{\omega_1}\alpha_{\omega_2}\cdots\alpha_{\omega_r}, \qquad \omega = (\omega_1, ..., \omega_r) \in \Delta,$$

where Δ is a complete set of representatives of equivalence classes of Γ. The solvability conditions (16) follow from Lemma 2.

References

1. W. H. Greub, "Multilinear Algebra," Springer-Verlag, Berlin/Heidelberg/New York, 1967.
2. H. W. Knobloch and F. Kappel, "Gewöhliche Differentialgleichungen," Teubner, Stuttgart, 1974.
3. P. Lancaster, "The Theory of Matrices," Academic Press, New York, 1969.
4. J. G. Malkin, "Theorie der Stabilität einer Bewegung," Oldenburg, München, 1959.
5. M. Marcus, "Finite Dimensional Multilinear Algebra," Part I, Dekker, New York, 1973.